Gravitation: A distortion gauge theory

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Gravitation is introduced as a gauge field that is invariant under the group of translations, the abelian subgroup of the Poincaré displacement group. The distortions in space created by a gravitational field can be compared to the distortions in a crystal characterized by its Burger's vector[7]. The space time frame is a flat space with no curvature and Minkowski metric of signature +1, -1,-1,-1. The starting point for the equations of motion and the gravitational field is a naturally occurring gauge invariant Lagrangian. The equations of motion in a central static field yield interesting results: a repulsive gravitational potential appears at short distance, black holes can have a stable and finite size, the expansion of the universe and its acceleration can be explained without recourse to a hypothetical dark energy.

CONTENTS

I. INTRODUCTION

After Einstein produced the famous General Relativity (GR) theory for gravitation, several theories were then also proposed to either improve, reinterpret or supplement GR. The aim of these efforts was to quantify gravitation and explain later discoveries like the expansion of the universe. Several of these theories are based on the non-zero torsion of the space time. Elie Cartan did develop a model where torsion would be generated by angular momentum [8, 9] [10]: Cartan associates to each closed infinitesimal contour a rotation (which expresses curvature) and a translation (which expresses torsion). In Cartan's mind the rotation can be represented by a vector and the translation by a torque. Tetrad formalism, teleparallel gravity, Weitzenböck and Möller theories are shown to be equivalent to GR in reference[12], and reference [18] shows that the Schwartzschild metric can have an interpretation of teleparallelism in the Pound-Rebka experiment. An Introduction to teleparallel gravity is given in reference [13] where curvature =0 and torsion is the gravitational field strength. In consequence, there are no geodesics in Teleparallel Gravity, only force equations quite analogous to the Lorentz force equation of electrodynamics. The authors expected this result by because, like electrodynamics, Teleparallel Gravity is also a gauge theory. Gauge theories for gravitation are treated in [11] in a formal mathematical frame, and in [12, 15] where a mathematical formalism in which torsion and curvature can be exchanged via a supergauge symmetry leads to the GR equations. Translation gauge potentials [19] meet Cartan's idea of the spin of matter being the source of torsion: The gauge gravitation theory based on the relativity and equivalence principles reformulated in fibre bundle terms is the gravitation theory with torsion whose source is the spin of matter. "Since transla-

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tion gauge potentials fail to be utilized for describing a gravitational field, a question on their physical meaning arises." "Therefore, translation gauge potentials may be responsible for weaker forces than gravity as discussed by some of the authors." Goldstonic supergravity, supergroups, supertransformations, superspace, supersymmetries, superfunctions etc. are examined in a mathematical frame in this paper. The expansion of the universe and its acceleration are explained in the framework of GR by the action of a phantom dark energy [5, 6] in the ΛCDM model.

In the present paper we do away with GR: the gauge invariance under translation of the Lagrangian is the starting point in a space with no curvature but only distortion which is a local translation field.

The space is flat, with Euclidian connexion and coordinates such that the covariant derivative is just the simple derivative, [3] [ch IV].

The Christoffel symbols are zero, thus the torsion tensor is also zero.

Preliminary about notation: In our flat Minkowskian space, the covariant and contravariant indices are the same:$X_\mu = X^\mu$. In order to use the Einstein convention for summing indices, the indices to be summed will be placed at the top and bottom: $X_\mu X^\mu = X^2$.

II. STARTING POINT, THE FIELD OF DISTORTIONS

Let us consider the Pound-Rebka experiment: it shows that a gravitational field can affect the time scale and create a distortion in space-time. The figure 1 shows the layout of the experiment: 1

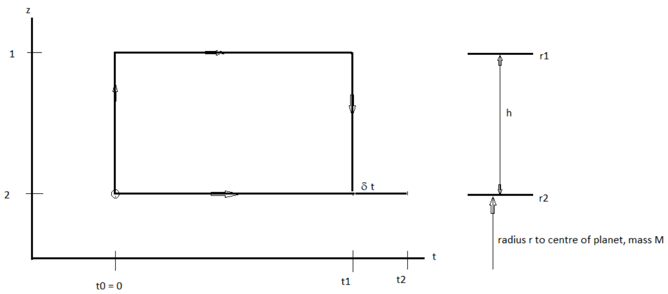

Figure 1. Pound-Rebka experiment

The frequency of an oscillator increases at point 1 compared to point 2. The period of an oscillation is thus shorter at point 1 than at point 2.
Let us write this period difference δt and $\nu_{1,2}$ the frequencies at point 1 and 2. The Pound-Rebka experiment result was: 1

$$\nu_2(1 - \frac{GM}{(r+h)c^2}) = \nu_1(1 - \frac{GM}{rc^2}) \tag{1}$$

with M the earth mass, G the gravitational constant, c the speed of light and $r_{1,2}$ the distance of points 1 and 2 to the center of Earth.

Let us write Φ the gravitational potential $\frac{GM}{rc^2}$ and $t_{1,2} = 1/\nu_{1,2}$ the oscillation periods. Then $\Phi_1 < \Phi_2$ and

$$\delta t = t_2 - t_1 = t_2\Phi_2 - t_1\Phi_1. \tag{2}$$

For $\delta t \ll t_1, t_2$ one has $\delta t = t_1(\Phi_1 - \Phi_2)$. This δt translation in time can be represented as a gauge field Φ acting on the a b c d contour:

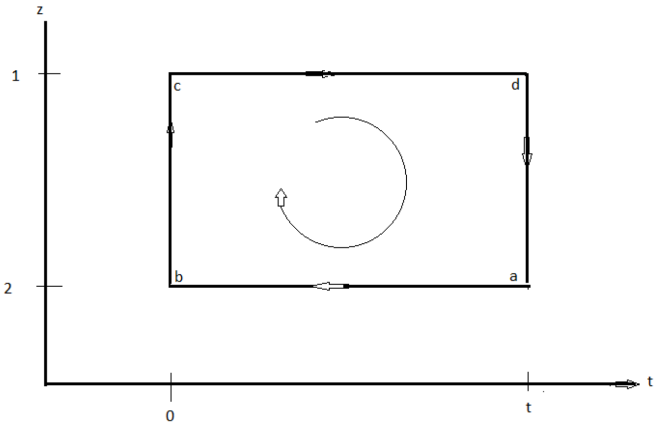

Figure 2. Gauge field induced time translation

$$\oint \Phi\, dl = \delta t = (t_b - t_a)\Phi_2 + (t_c - t_b)\Phi_{21} + (t_d - t_c)\Phi_1 + (t_a - t_d)\Phi_{21}. \tag{3}$$

$$\Rightarrow \delta t = t\Phi_1 - t\Phi_2, \tag{4}$$

with $(t_d - t_c) = -(t_b - t_a) = t$ and $t_c = t_b$ and $t_a = t_d$.

The gravitational potential is a gauge field that creates a distortion of time: time is contracted at point 1 compared to point 2 and in closing the contour, there is a gap δt that remains open. The contour is distorted by the gravitational field. Since $dt_1 < dt_2$, and there is no change in x y z dimensions, the speed of light must then be higher at point 1 than at point 2.

The field of distortions-translations

Consider the action of a field Γ that can deviate a displacement vector x in x': The vector $\underline{x' - x}$ is $\overrightarrow{\xi}$ a translation added to x generated by the field Γ.
For an infinitesimal displacement dx, dx' will be: $dx'_\mu = dx_\mu + d\xi_\mu$ with $d\xi_\mu = \Gamma^\nu_\mu dx_\nu$

Under the action of the gauge field Γ , dx_μ becomes $dx'_\mu = dx_\mu + \Gamma^\nu_\mu\, dx\nu$

For instance in the Pound Rebka experiment, in the presence of a gravitational field a time lapse dt is increased by Φdt ; $dt' = dt + \Phi dt$ with $\Phi = \Gamma^0_0 = \frac{GM}{rc^2}$. The index 0 is for time.

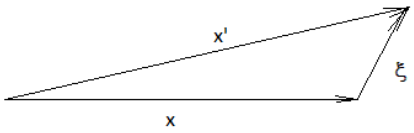

Figure 3. local ξ translation

III. EQUATIONS OF MOTION IN THE GAUGE FIELD Γ_μ^ν

The trajectory of a body between two points A and B is the path where the integral of the action dS is minimal, with $dS = p^\mu dx_\mu$, and p^μ being the impulsion of the body.

The occurrence of a field Γ_μ^ν will influence dx_μ and generate the equations of motion of a body in a gravitational field.

Let us consider the Lagrangian equation of motion, from a geometrical point of view:
A small deviation from the trajectory between the two points A and B will not change the action to the first order:

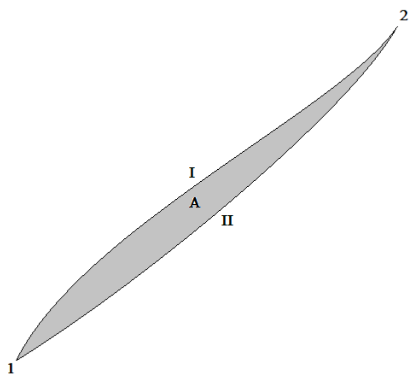

Figure 4. Action variation

$$\int_I p^\mu \, dx_\mu = \int_{II} p^\mu \, dx_\mu$$

or: $\oint_{I,-II} p^\mu \, dx_\mu = 0$

Which is equivalent to: $\iint_A (\partial_\mu p^\nu - \partial_\nu p^\mu) dS = 0$

Thus $\partial_\mu p^\nu - \partial_\nu p^\mu = 0$

Multiplying by $v_\mu = \frac{dx_\mu}{dt}$, one gets: $\frac{dx_\mu}{dt}\frac{\partial p^\nu}{\partial x_\mu} - v_\mu \partial_\nu p^\mu = 0$

Since $\partial_\nu v_\mu = 0$ and $\frac{dx_\mu}{dt}\frac{\partial p^\nu}{\partial x_\mu} = \frac{d}{dt}p^\nu$

We have: $\frac{d}{dt}p^\nu - \frac{\partial}{\partial x_\nu}(v_\mu p^\mu) = 0$

But $v_\mu p^\mu$ is $\mathcal{L}$ the Lagrangian, where:

$S = \int p^\mu \, dx_\mu = \int p^\mu v_\mu \, dt = \int \mathcal{L} \, dt$

Then $\frac{d}{dt}p^\nu - \frac{\partial}{\partial x_\nu}\mathcal{L} = 0$ or $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_\nu} - \frac{\partial}{\partial x_\nu}\mathcal{L} = 0$

With $p^\nu = \frac{\partial \mathcal{L}}{\partial v_\nu}$

If a field Γ_μ^ν is introduced with $dx'_\mu = dx_\mu + \Gamma_\mu^\nu \, dx_\nu$

Then

$$\begin{aligned} S &= \int p^\mu \, dx'_\mu \\ &= \int p^\mu \, (dx_\mu + \Gamma_\mu^\nu \, dx\nu) \\ &= \int p^\mu (\delta_\mu^\nu + \Gamma_\mu^\nu) \, dx_\nu \end{aligned}$$

Thus $S = \int p^\mu (\delta_\mu^\nu + \Gamma_\mu^\nu) v_\nu \, dt = \int \mathcal{L} \, dt$

Which implies

$$\mathcal{L} = p^\mu (\delta_\mu^\nu + \Gamma_\mu^\nu) v_\nu \tag{5}$$

Equations of motion

1st term:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_\nu} = \frac{d}{dt}(p^\nu + \Gamma_\mu^\nu p^\mu) = \frac{d}{dt}p^\nu + p^\mu \frac{d}{dt}\Gamma_\mu^\nu + \Gamma_\mu^\nu \frac{d}{dt}p^\mu \tag{6}$$

$$= \frac{d}{dt}p^\nu + p^\mu v_\rho \frac{\partial \Gamma_\mu^\nu}{\partial x_\rho} + \Gamma_\mu^\nu v_\rho \frac{\partial p^\mu}{\partial x_\rho} \tag{7}$$

2d term:

$$\frac{\partial \mathcal{L}}{\partial x_\nu} = \frac{\partial}{\partial x_\nu}(p^\rho v_\rho) + \frac{\partial}{\partial x_\nu}(p^\mu \Gamma_\mu^\rho v_\rho) \tag{8}$$

At this stage, we simplify the notation: $\partial_\nu \equiv \frac{\partial}{\partial x_\nu}$
Then, since $\partial_\nu v_\rho = 0$

$$\partial_\nu \mathcal{L} = v_\rho \partial_\nu p^\rho + p^\mu v_\rho \partial_\nu \Gamma_\mu^\rho + v_\rho \Gamma_\mu^\rho \partial_\nu p^\mu \tag{9}$$

Assembling the two terms in the Lagrange equation we get:

$$\begin{aligned} v_\mu \partial_\mu p^\nu + \Gamma_\mu^{nu} v_\rho \partial_\rho p^\mu = p^\mu v_\rho (\partial_\nu \Gamma_\mu^\rho - \partial_\rho \Gamma_\mu^\nu) \\ + v_\rho \Gamma_\mu^\rho \partial_\nu p^\mu + v_\mu \partial_\nu p^\mu \end{aligned} \tag{10}$$

Or:

$$\begin{aligned} v_\mu (\partial_\mu p^\nu + \partial_\nu p^\mu) + v_\rho (\Gamma_\mu^\nu \partial_\rho p^\mu + \Gamma_\mu^\rho \partial_\nu p^\mu) \\ = p^\mu v_\rho (\partial_\nu \Gamma_\mu^\rho - \partial_\rho \Gamma_\mu^\nu) \end{aligned} \tag{11}$$

Or:

$$\begin{aligned} v_\mu (\partial_\mu p^\nu - \partial_\nu p^\mu) = v_\rho [\partial_\nu (p^\mu \Gamma_\mu^\rho) - \partial_\rho (p^\mu \Gamma_\mu^\nu)] \\ \Leftrightarrow v_\rho (\partial_\rho p^\nu - \partial_\nu p^\rho) = v_\rho [\partial_\nu (p^\mu \Gamma_\mu^\rho) - \partial_\rho (p^\mu \Gamma_\mu^\nu)] \end{aligned} \tag{12}$$

The equation 12 is satisfied if

$$\partial_\rho (p^\nu + p^\mu \Gamma_\mu^\nu) - \partial_\nu (p^\rho + p^\mu \Gamma_\mu^\rho) = 0 \tag{13}$$

Temporal and spatial fields

Equation 12 can be rewritten, if we define $\gamma^\nu \equiv p^\mu \Gamma^\nu_\mu$

$$v_\rho \partial_\rho p^\nu - v_\rho \partial_\nu p^\rho = v_\rho(\partial_\nu \gamma^\rho - \partial_\rho \gamma^\nu) \tag{14}$$

Or:

$$\frac{d}{dt} p^\nu = \frac{\partial \mathcal{L}}{\partial x_\nu} + v_\rho(\partial_\nu \gamma^\rho - \partial_\rho \gamma^\nu) \tag{15}$$

If no extra fields than the gravitational field are considered, $\frac{\partial \mathcal{L}}{\partial x_\nu}$ can be ignored and
For $\nu = 0$ one gets:

$$\frac{d}{dt} p^0 = v_i(\partial_t \gamma^i - \partial_i \gamma^0) \ ; \ i = 1,2,3 \tag{16}$$

Or if we define $E_i \equiv \partial_i \gamma^0 - \partial_t \gamma^i$
We get:

$$\frac{d}{dt} p^0 = -v_i E^i \tag{17}$$

For $\nu = 1,2,3$ noted $\nu = i$:

$$\frac{d}{dt} p^i = v_\rho(\partial_i \gamma^\rho - \partial_\rho \gamma^i) \ ; \ \rho = 0,1,2,3 \tag{18}$$

For $\rho = 0$: $v_0 \equiv 1$ and $1(\partial_i \gamma^0 - \partial_t \gamma^i) = E^i$ again.
For $\rho = 1,2,3$ noted $\rho = j$:
One has:

$$v_j(\partial_i \gamma^j - \partial_j \gamma^i) = \vec{v} \times \overrightarrow{rot\vec{\gamma}} \tag{19}$$

If we define $\vec{H} \equiv \overrightarrow{rot\vec{\gamma}}$
We finally get:

$$\frac{d}{dt} \vec{p} = \vec{E} + (\vec{v} \times \vec{H}) \tag{20}$$

IV. SYMMETRY OF Γ^ν_μ AND GAUGE INVARIANCE

Considering that dx is distorted by the gauge field into dX, then $\frac{d}{dx}$ will be changed in $\frac{d}{dX}$ when the reference frame is changed. For instance in the Pound-Rebka experiment, position 1 and position 2. This derivative applies to whatever object, scalar, vector and tensor.
If $dX_\mu = dx_\mu + \Gamma^\nu_\mu dx_\nu$, then $\frac{d}{dX_\mu} = (\delta^\nu_\mu + \Gamma^\nu_\mu)^{-1} \frac{d}{dx_\nu} \simeq \frac{d}{dx_\mu} - \Gamma^\nu_\mu \frac{d}{dx_\nu}$ for $\Gamma^\nu_\mu \ll 1$
For the distant observer, $\Gamma^\nu_\mu = 0$ and $\frac{d}{dX}$ becomes $\frac{d}{dx}$.
Note that the metric being Euclidian / Minkowskian, the Christoffel symbols are zero and the space has no curvature neither torsion, only distortion. Again with null Christoffel symbols the covariant derivative is just $\frac{d}{dx}$ that applies to whatever object, scalar, vector and tensor.

Symmetry of μ and ν in Γ^ν_μ

The symmetry of μ and ν in Γ^ν_μ ensures that there will be no rotation but only distortion of space, as shown in the following figure and which can be proved easily.
Thus Γ^ν_μ will have 10 components.
And Γ^ν_μ will be identified with the graviton that has spin 2 (symmetric tensor potential) [2]
On the following figure, the symmetrical $\Gamma^2_1 = \Gamma^1_2$ creates a distortion in the 1,2 plane:

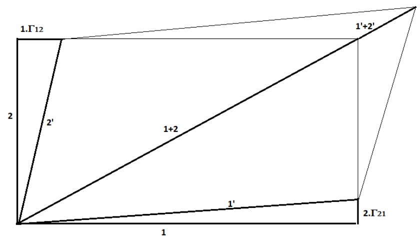

Figure 5. distortion

$\vec{1'} + \vec{2'} = \vec{1} + \vec{2}\gamma^2_1 + \vec{2} + \vec{1}\Gamma^1_2 = (\vec{1} + \vec{2})(1 + \Gamma^2_1)$
If $\Gamma^2_1 = -\Gamma^1_2$ then we would have a rotation instead of a distortion, and Γ^0_0 would then also be $= 0$

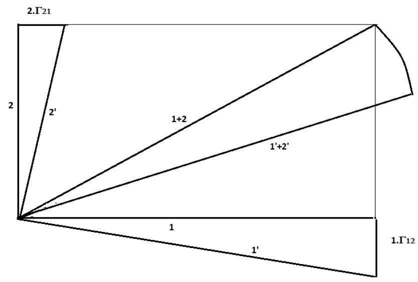

Figure 6. rotation

$\vec{1'} + \vec{2'} = \vec{1} + \vec{2}\Gamma^2_1 + \vec{2} + \vec{1}\Gamma^1_2 = \vec{1} + \vec{2}\Gamma^2_1 + \vec{2} - \vec{1}\Gamma^2_1$

Gauge invariance

The action $S = \int p^\mu(\delta^\mu_\nu + \Gamma^\mu_\nu)\, dx_\nu$ is invariant under translation.
Indeed if x_ν is transported to $x_\nu + G_\nu$ with G_ν a constant vector, then $dG_\nu = 0$ and $dx_\nu \to dx_\nu + dG_\nu = dx_\nu$.
Concerning the Γ^μ_ν fields: The addition of a gradient $\partial_\nu G$ to $p^\mu \Gamma^\nu_\mu$ leaves the equation of motion 13 unchanged:

$$p^\mu \Gamma^\nu_\mu \to p^\mu \Gamma^\nu_\mu + \partial_\nu G \tag{21}$$

Then $\partial_\rho(p_\nu + p_\mu \Gamma^\mu_\nu + \partial_\nu G) - \partial_\nu(p_\rho + p_\mu \Gamma^\mu_\rho + \partial_\rho G) = 0$ since $\partial_\rho \partial_\nu G - \partial_\nu \partial_\rho G = 0$
It will be seen later the field of forces can be expressed

by: $F^{\rho}_{\mu\nu}= (\partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \Gamma^{\nu}_{\mu})$ and this field is invariant if:

$$\Gamma^{\nu}_{\mu} \to \Gamma^{\nu}_{\mu} + \partial_\nu B^{\mu} \tag{22}$$

with $\overrightarrow{B}$ = an arbitrary vector field. 21 and 22 can be shown to be equivalent if $\overrightarrow{B} = \kappa \overrightarrow{p}$

V. THE EQUATIONS OF FIELD

We look for an action that is a scalar, gauge invariant and that includes only the Γ^{ν}_{μ} terms: This action is noted S_f.
Let us start from 15: $\frac{d}{dt}p^{\nu} = \frac{\partial \mathcal{L}}{\partial x_\nu} + v_\rho(\partial_\nu \gamma^{\rho} - \partial_\rho \gamma^{\nu})$

Expanding and neglecting $\frac{\partial \mathcal{L}}{\partial x_\nu}$ (no other potential than gravitational), we get:

$$\frac{d}{dt}p^{\nu} = v_\rho p^{\mu} \underbrace{(\partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \Gamma^{\nu}_{\mu})}_{I} + v_\rho \underbrace{(\Gamma^{\rho}_{\mu} \partial_\nu p^{\mu} - \Gamma^{\nu}_{\mu} \partial_\rho p^{\mu})}_{II} \tag{23}$$

We look at a gauge invariant scalar term that depends only on the field Γ^{ν}_{μ}:
It is: $(\partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \Gamma^{\nu}_{\mu})^2$
We define the action of the field as:

$$Sf = \alpha \int (\partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \Gamma^{\nu}_{\mu})(\partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \Gamma^{\nu}_{\mu})\, dV dt \tag{24}$$

With α an arbitrary constant, V is the spatial volume. The action is now completed with the field – matter interaction:

$$S = \int p^{\mu} \Gamma^{\nu}_{\mu} v_\nu \, dV dt + \alpha \int (\partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \Gamma^{\nu}_{\mu})(\partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \Gamma^{\nu}_{\mu})\, dV dt \tag{25}$$

Where p^{μ} stands for the density of impulsion.
How does the action vary under a variation of the potential Γ ?

$$\delta S = \int p^{\mu} v_\nu \, \delta \Gamma^{\nu}_{\mu} + \alpha\, \delta(\partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \Gamma^{\nu}_{\mu})^2 \, dV dt \tag{26}$$

$$\delta S = \int p^{\mu} v_\nu \, \delta \Gamma^{\nu}_{\mu} + 2\alpha(\partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \Gamma^{\nu}_{\mu})\; \delta(\partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \Gamma^{\nu}_{\mu})\, dV dt \tag{27}$$

$$\delta S = \int p^{\mu} v_\nu \, \delta \Gamma^{\nu}_{\mu} + 2\alpha[(\partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \Gamma^{\nu}_{\mu})\, \partial_\nu \delta \Gamma^{\rho}_{\mu} - (\partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \Gamma^{\nu}_{\mu})\, \partial_\rho \delta \Gamma^{\nu}_{\mu}]\, dV dt \tag{28}$$

We have swapped the ∂ and δ, and swapping the ρ and ν indices we get:

$$\delta S = \int p^{\mu} v_\nu \, \delta \Gamma^{\nu}_{\mu} + 4\alpha(\partial_\rho \Gamma^{\nu}_{\mu} - \partial_\nu \Gamma^{\rho}_{\mu})\, \partial_\rho \delta \Gamma^{\nu}_{\mu}\, dV dt \tag{29}$$

The term in 4α is integrated by parts:

$$\begin{aligned} \int (\partial_\rho \Gamma^{\nu}_{\mu} - \partial_\nu \Gamma^{\rho}_{\mu})\, \partial_\rho \delta \Gamma^{\nu}_{\mu}\, dV dt &= -\int \partial_\rho(\partial_\rho \Gamma^{\nu}_{\mu} - \partial_\nu \Gamma^{\rho}_{\mu})\, \delta \Gamma^{\nu}_{\mu}\, dV dt \\ &+ \int (\partial_\rho \Gamma^{\nu}_{\mu} - \partial_\nu \Gamma^{\rho}_{\mu})\, \delta \Gamma^{\nu}_{\mu}\, dS_\rho \end{aligned} \tag{30}$$

The term $\int (\partial_\rho \Gamma^{\nu}_{\mu} - \partial_\nu \Gamma^{\rho}_{\mu})\, \delta \Gamma^{\nu}_{\mu}\, dS_\rho = 0$ because $\delta \Gamma^{\nu}_{\mu} = 0$ on the time limits and $(\partial_\rho \Gamma^{\nu}_{\mu} - \partial_\nu \Gamma^{\rho}_{\mu}) = 0$ at ∞. The field strength is 0 on the border at ∞

Thus we obtain:

$$\delta S = \int p^{\mu} v_\nu \, \delta \Gamma^{\nu}_{\mu} - 4\alpha \partial_\rho(\partial_\rho \Gamma^{\nu}_{\mu} - \partial_\nu \Gamma^{\rho}_{\mu})\, \delta \Gamma^{\nu}_{\mu}\, dV dt \tag{31}$$

By cancelling the variation of S we have:

$$p^{\mu} v_\nu - 4\alpha \partial_\rho(\partial_\rho \Gamma^{\nu}_{\mu} - \partial_\nu \Gamma^{\rho}_{\mu}) = 0 \tag{32}$$

A corollary of eq 32: the divergence of the energy impulsion tensor $(p^{\mu} v_\nu)$ is equal to zero.

$$\begin{aligned} \partial_\nu(p^{\mu} v_\nu) &= 4\alpha(\partial_\nu \partial_\rho \partial_\rho \Gamma^{\nu}_{\mu} - \partial_\nu \partial_\nu \partial_\rho \Gamma^{\rho}_{\mu}) \\ &= 4\alpha(\partial_\rho \partial_\nu \partial_\nu \Gamma^{\rho}_{\mu} - \partial_\rho \partial_\nu \partial_\nu \Gamma^{\rho}_{\mu}) = 0 \end{aligned} \tag{33}$$

Now we evaluate 4α:
If the source current $p^{\mu} v_\nu$ is generated by a mass with rest density ρ : $p^{\mu} v_\nu = \rho\, \delta^{\mu}_{0} \delta^{0}_{\nu} c^2$
And from 32 we get:

$$\rho\, c^2 = 4\alpha[\partial_\rho \partial_\rho \Gamma^{0}_{0} - \partial_\rho \partial_t \Gamma^{0}_{0}] \tag{34}$$

With the mass density ρ at rest the field Γ^{0}_{0} must be static: $\partial_t \Gamma^{0}_{0} = 0$
And $\rho\, c^2 = 4\alpha\, \Delta\, \Gamma^{0}_{0}$ with Δ the Laplacian
Thus $\Gamma^{0}_{0} = \int \frac{c^2 \rho}{4\alpha r}\, dV$ and since $\Gamma^{0}_{0} = \frac{GM}{rc^2}$,
We get:

$$4\alpha = \frac{c^4}{G}\ [m \frac{kg}{s^2}] \tag{35}$$

And eq 32 can be rewritten:

$$\partial_\rho(\partial_\rho \Gamma^{\nu}_{\mu} - \partial_\nu \Gamma^{\rho}_{\mu}) = \frac{G}{c^4} p_\mu v^{\nu}\ [m^{-2}] \tag{36}$$

Where p_μ is the density of impulsion.
Depending on the μ, ν values we have for the equations of the field:
1) μ, ν =0; $p_0 v_0 = \rho c^2$

$$\frac{G}{c^2}\rho = \partial_\lambda \partial_\lambda \Gamma^{0}_{0} - \partial_\lambda \partial_t \Gamma^{\lambda}_{0}\ [m^{-2}] \tag{37}$$

If the field is static, we get $\Delta\Gamma_0^0 = \frac{G}{c^2}\rho \to \Gamma_0^0 = \frac{GM}{rc^2}$
With $M = \int \rho\, dV$
The equations of motion in a central static field are treated in the next chapter.
2) If $\mu, \nu \neq 0$ noted i, j then for $v \ll c$, $p \simeq \rho v$ and we get:

$$\frac{G}{c^4}\rho v_i v_j = \partial_\lambda(\partial_\lambda \Gamma_i^j - \partial_i \Gamma_j^\lambda)\,[m^{-2}] \tag{38}$$

This is made symmetrical in i and j because $\Gamma_i^j = \Gamma_j^i$ and by introducing the "Lorentz" condition: $\partial_\lambda \Gamma_j^\lambda = 0$

Gravitational waves

Equation 75 can be rewritten assuming a "Lorentz condition": $\partial_\lambda \Gamma_j^\lambda = 0$

$$\partial_\rho \partial_\rho \Gamma_\nu^\mu = \frac{G}{c^4} p^\mu v_\nu \tag{39}$$

In the vacuum:

$$\partial_\rho \partial_\rho \Gamma_\nu^\mu = 0 = (\frac{1}{c^2}\partial_t \partial_t - \partial_i \partial_i)\Gamma_\nu^\mu$$

And the field Γ_ν^μ (the massless graviton) is a wave propagating at speed of light.

VI. THE TEST OF DEFLECTION OF LIGHT BY THE SUN

Scheme:
The gravitational potential of the sun is central and static:

$$\Gamma_\nu^\mu = \Gamma_0^0\, \delta_0^\mu\, \delta_\nu^0\, \Gamma_0^0 \equiv \Gamma \tag{40}$$

The impulsion of the photon is $\vec{p} = \hbar \vec{k}$

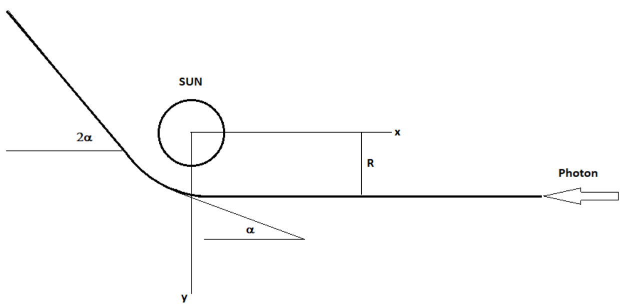

Figure 7. Deflection of light

And $\vec{k} = (\omega, k_0\infty, 0, 0) = (\omega, \omega, 0, 0)$ $\vec{v} = (1, -1, 0, 0)$ with c=1

With the symbol $\dot{k} = \frac{d}{dt}k$ the equation 15 gives:

$$\dot{k}_\nu = v_\rho\,[\partial_\nu(k^\mu \Gamma_\mu^\rho) - \partial_\rho(k^\mu \Gamma_\mu^\nu)] \tag{41}$$

With Γ_μ^ν defined by 40, this gives:

$$\dot{k}_\nu = v_0 \partial_\nu(k_0\Gamma) - v_\rho \partial_\rho(k_\mu \Gamma_\nu^\mu) \tag{42}$$

For $\dot{k}_0$: $\dot{k}_0 = v_0\partial_t(k_0\Gamma) - v_\rho\partial_\rho(k_0\Gamma)$; $\rho = (1, 2, 3)$

$$\dot{k}_0 = \partial_t(k_0\Gamma) + \partial_1(k_0\Gamma) = \Gamma\partial_t k_0 + k_0\partial_t\Gamma + \Gamma\partial_1 k_0 + k_0\partial_0\Gamma$$

Since $\partial_t\Gamma = 0$, $\dot{k}_0 = \Gamma(\partial_t k_0 + \partial_1 k_0) + k_0\partial_1\Gamma$
But $x = x_\infty - ct = x_\infty - t$(with c=1)
$\to \partial_t = -\partial_1$ Note: $\partial_1 \equiv \partial_x$
Thus

$$\dot{k}_0 = k_0\partial_1\Gamma \tag{43}$$

For $\dot{k}_i$: $\dot{k}_i = v_0\partial_i(k_0\Gamma) - v_\rho\partial_\rho(k_\mu\gamma_\mu^i) = \partial_i(k_0\Gamma)$
With $v_0 = 1$ and $\Gamma_\mu^i = 0\,; i \neq 0$
And $\dot{k}_2 = \partial_2(k_0\Gamma) = \partial_y(k_0\Gamma) = k_0\partial_2\Gamma$ since $\partial_2 k_0 = 0$
We combine: $\dot{k}_0 = k_0\partial_1\Gamma$ and get:

$$\dot{k}_2 = k_0\partial_2\Gamma \tag{44}$$

Derivating $\dot{k}_2$ to time we get: $\ddot{k}_2 = \partial_2(\dot{\overline{k_0\Gamma}}) = (k_0\dot{\partial_2}\Gamma) = \dot{k}_0\partial_2\Gamma + k_0\partial_2\dot{\Gamma} \to \ddot{k}_2 = k_0\partial_1\Gamma\partial_2\Gamma + k_0\partial_2(v_j\partial_j\Gamma)$
The first term is in Γ^2 and can be neglected versus Γ ($\Gamma \approx 10^{-6}$ at surface of the sun)
Thus $\ddot{k}_2 = k_0\partial_2(v_0\partial_t\Gamma + v_1\partial_1\Gamma)$, with $v_0 = 1$, $v_1 = -1$ and $\partial_t = -\partial_1$
This leads to:

$$\ddot{k}_2 = k_0\partial_2(-\partial_1\Gamma - \partial_1\Gamma) = -2k_0\partial_1\partial_2\Gamma \tag{45}$$

Since $\Gamma = \frac{GM}{r} = \frac{GM}{\sqrt{x^2+y^2}}$ with c=1

$$\partial_2\Gamma = \frac{-yGM}{(x^2+y^2)^{3/2}}$$

$$\partial_1\partial_2\Gamma = \frac{3xyGM}{(x^2+y^2)^{5/2}}$$

then

$$\ddot{k}_2 = -6k_0\frac{xyGM}{(x^2+y^2)^{5/2}} \tag{46}$$

With $\ddot{k}_2 = \frac{d}{dt}\dot{k}_2 = \frac{dx}{dt}\frac{d\dot{k}_2}{dx}$ and $\frac{dx}{dt} = -1$
We get $\dot{k}_2 = -\int \ddot{k}_2\, dx$
Thus

$$\dot{k}_2 = 6k_0 GM \int \frac{xy}{(x^2+y^2)^{5/2}}\, dx \tag{47}$$

Integrating again, with $dx = -dt$

$$k_2 = 2k_0 GMy \int_{\infty}^{0} \frac{1}{(x^2+y^2)^{3/2}}\, dx = -2k_0 \frac{GMy}{y^2} \quad (48)$$

Thus $k_2 = -2k_0 \frac{GM}{y}$
For $y = R$,

$$\frac{k_2}{k_0} = \frac{-2GM}{R} = \tan(\alpha) \simeq \alpha \quad (49)$$

with c=1

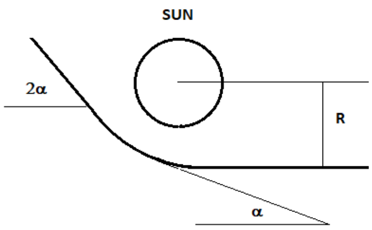

Figure 8. Total deviation

The total deviation is $2\alpha = \frac{4GM}{Rc^2}$
This corresponds to the measured value.

VII. EQUATION OF MOTION IN A CENTRAL STATIC FIELD

The field $\Gamma_i^j = \Gamma_0^0\, \delta_0^j\, \delta_i^0$, $\Gamma_0^0 \equiv \Gamma \quad \partial_t \Gamma = 0 \quad \dot{\Gamma} = v_j \partial_j \Gamma$, and $\Gamma = \frac{GM}{rc^2}$
Starting from 15 :

$$\dot{p_\nu} = v_\rho[\partial_\nu(p^\mu \Gamma_\mu^\rho) - \partial_\rho(p^\mu \Gamma_\mu^\nu)]$$

For i=1,2,3 one has:

$$\dot{p_i} = v_0 \partial_i(p_0\Gamma) - v_j\partial_j(p_0\Gamma_0^i) = \partial_i(p_0\Gamma) \quad (50)$$

For i=0:

$$\dot{p_o} = v_0\partial_t(p_0\Gamma) - v_i\partial_i(p_0\Gamma) \quad (51)$$

Thus:

$$\dot{p_i} = \partial_i(p_0\Gamma) = p_0\partial_i\Gamma + \Gamma\partial_i p_0 \quad (52)$$

And:

$$\dot{p_0} = -v_i\partial_i(p_0\Gamma) + \Gamma\partial_t p_0 \quad (53)$$

We now calculate $\ddot{p_i}$:

$$\ddot{p_i} = (p_0\partial_i\Gamma \dot{+} \Gamma\partial_i p_0) = \dot{p_0}\partial_i\Gamma + p_0\dot{\partial_i\Gamma} + \dot{\Gamma}\partial_i p_0 + \Gamma\partial_i\dot{p_0} \quad (54)$$

Replace $\dot{p_0}$ in 54:

$$\ddot{p_i} = \underbrace{(-v_j\partial_j(p_0\Gamma) + \Gamma\partial_t p_0)\partial_i\Gamma}_{I} + \underbrace{(p_0\partial_i(v_j\partial_j\Gamma)}_{II} + \underbrace{v_j\partial_j\Gamma\partial_i p_0}_{III} + \underbrace{\Gamma\partial_i(-v_j\partial_j(p_0\Gamma) + \Gamma\partial_t p_0)}_{IV} \quad (55)$$

The terms I+IV give: $\partial_i[\Gamma(-v_j\partial_j(p_0\Gamma) + \Gamma^2\partial_t p_0]$
The terms II + III give: $\partial_i[p_0 v_j\partial_j\Gamma]$
Thus

$$\ddot{p_i} = \partial_i[p_0 v_j\partial_j\Gamma + \Gamma^2\partial_t p_0 - \Gamma v_j\partial_j(p_0\Gamma)] \quad (56)$$
$$= \partial_i[p_0 v_j\partial_j\Gamma + \Gamma^2\partial_t p_0 - \Gamma v_j(p_0\partial_j\Gamma + \Gamma\partial_j p_0)] \quad (57)$$
$$= \partial_i[(p_0 v_j\partial_j\Gamma)(1-\Gamma) + \Gamma^2(\partial_t p_0 - v_j\partial_j p_0)] \quad (58)$$

The term $\partial_t p_0 - v_j\partial_j p_0 = \dot{p_0}$
But eq 53 gives:

$$\dot{p_0} = \Gamma\partial_t p_0 - \Gamma v_i\partial_i p_0 - v_i p_0\partial_i\Gamma = \Gamma\dot{p_0} - v_i p_0\partial_i\Gamma \quad (59)$$

Then $\dot{p_0}(1-\Gamma) = -p_0 v_i\partial_i\Gamma$ or $\dot{p_0} = \frac{-p_0 v_i\partial_i\Gamma}{1-\Gamma}$

Replacing in eq 58 we get:

$$\ddot{p_i} = \partial_i\left[(p_0 v_j\partial_j\Gamma)(1-\Gamma) + \frac{\Gamma^2}{1-\Gamma}(-p_0 v_j\partial_j\Gamma)\right] \quad (60)$$

Or:

$$\ddot{p_i} = \partial_i\left[(p_0 v_j\partial_j\Gamma)\frac{1-2\Gamma}{1-\Gamma}\right] \quad (61)$$

Non Relativistic Equations of motion :

For $v \ll c$ and no external field, $p_0 \simeq mc^2$ and $p_j \simeq mv_j$

$$\ddot{p_i} = mc^2 v_j\partial_i\left[(1-\Gamma)\partial_j\Gamma - \frac{\Gamma^2}{1-\Gamma}\partial_j\Gamma\right] = mv_j\partial_i\left[\frac{1-2\Gamma}{1-\Gamma}\partial_j\Gamma\right] \quad (62)$$

Or:

$$m\ddot{v_i} = mc^2 v_j\left[\frac{1-2\Gamma}{1-\Gamma}\partial_i\partial_j\Gamma - \frac{\partial_i\Gamma\partial_j\Gamma}{1-\Gamma^2}\right] \quad (63)$$

Resulting in:

$$\frac{\ddot{v_i}}{c^2} = v_j\frac{1}{(1-\Gamma)^2}[(1-3\Gamma+2\Gamma^2)\partial_i\partial_j\Gamma - \partial_i\Gamma\partial_j\Gamma] \quad (64)$$

Equations of motion in polar coordinates r, φ :

Let us now evaluate the equation of motion 64 of a body in the central static gravitational field Γ , with $\Gamma = \frac{GM}{c^2 r}$ and $\frac{\partial \Gamma}{\partial \varphi} = 0$ in polar coordinates r, φ:

In the referential $\vec{e}$, $\vec{n}$: $\vec{n} = \frac{d\vec{e}}{d\varphi}$ and $\vec{e} = -\frac{d\vec{n}}{d\varphi}$
Thus $\dot{\vec{e}} = \vec{n}\dot{\varphi}$ and $\dot{\vec{n}} = -\vec{e}\dot{\varphi}$

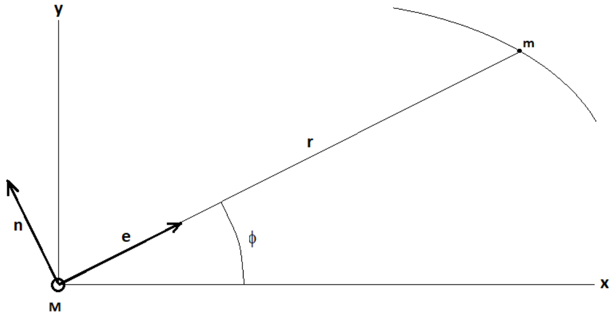

Figure 9. polar coordinates

The velocity $\vec{v} = \dot{r}\vec{e} + r\dot{\varphi}\vec{n}$
The gradient of a scalar f writes: $\vec{\partial}$f= $\frac{\partial f}{\partial r}\vec{e} + 1/r\frac{\partial f}{\partial \varphi}\vec{n}$
Developing the equation 64, we have two terms:
Term I = $\underbrace{(1 - 3\Gamma + 2\Gamma^2) v_j \partial_j \partial_i \Gamma)}_{I}$
Term II =$\underbrace{(-v_j \partial_j \Gamma \partial_i \Gamma)}_{II}$
Developing each term I, II: (we leave the arrow on top of e and n for easier identification)
Term I: calculation of $v_j \partial_j \partial_i \Gamma$

$$v_j \partial_j \partial_i \Gamma = (\dot{r}\vec{e_j} + r\dot{\varphi}\vec{n_j} \,|\, \frac{2GM}{c^2 r^3}\vec{e_j}\vec{e_i} - \frac{GM}{c^2 r^3}\vec{n_j}\vec{n_i}) \quad (65)$$

Indeed:

$$\partial_j \partial_i \Gamma = \frac{\partial(\partial_i \Gamma)}{\partial r}\vec{e_j} + 1/r\frac{\partial(\partial_i \Gamma)}{\partial \varphi}\vec{n_j} \quad (66)$$

$$= \partial_r(\frac{-GM}{c^2 r^2}\vec{e_i})\vec{e_j} + 1/r\,\partial_\varphi(\frac{-GM}{c^2 r^2}\vec{e_i})\vec{n_j} \quad (67)$$

$$= \frac{2GM}{c^2 r^3}\vec{e_i}\vec{e_j} - \frac{GM}{c^2 r^2}(\partial_r \vec{e_i})\vec{e_j} + 1/r\left(\partial_\varphi(\frac{-GM}{c^2 r^2})\vec{e_i}\vec{n_j} - \frac{GM}{c^2 r^2}(\partial_\varphi \vec{e_i})\vec{n_j}\right) \quad (68)$$

$$= \frac{2GM}{c^2 r^3}\vec{e_i}\vec{e_j} - \frac{GM}{c^2 r^3}\vec{n_i}\vec{n_j} \quad (69)$$

Then

$$v_j \partial_j \partial_i \Gamma = \dot{r}\frac{2GM}{c^2 r^3}\vec{e_j}\vec{e_j}\vec{e_i} - \dot{r}\frac{GM}{c^2 r^3}\vec{e_j}\vec{n_j}\vec{n_i} + r\dot{\varphi}\frac{2GM}{c^2 r^3}\vec{n_j}\vec{e_j}\vec{e_i} - r\dot{\varphi}\frac{GM}{c^2 r^3}\vec{n_j}\vec{n_j}\vec{n_i} \quad (70)$$

Since $(\vec{e}|\vec{e}) = (\vec{n}|\vec{n}) = 1$ and $(\vec{e}|\vec{n}) = 0$ we get:

$$v_j \partial_j \partial_i \Gamma = \frac{2\dot{r}GM}{c^2 r^3}\vec{e_i} - \dot{\varphi}\frac{GM}{c^2 r^2}\vec{n_i}$$

And

$$(1 - 3\Gamma + 2\Gamma^2)\, v_j \partial_j \partial_i \Gamma = (2\dot{r}\frac{GM}{c^2 r^3} - 3\frac{GM}{c^2 r}\frac{2\dot{r}GM}{c^2 r^3} + 2\frac{G^2M^2}{c^4 r^2}\frac{2\dot{r}GM}{c^2 r^3})\vec{e_i} + (-\dot{\varphi}\frac{GM}{c^2 r^2} + 3\frac{GM}{c^2 r}\dot{\varphi}\frac{GM}{c^2 r^2} - 2\frac{G^2M^2}{c^4 r^2}\dot{\varphi}\frac{GM}{c^2 r^2})\vec{n_i} \quad (71)$$

Term II:

$$(-v_j \partial_j \Gamma \partial_i \Gamma) = -(\dot{r}\vec{e_j} + r\dot{\varphi}\vec{n_j} \,|\, \frac{-GM}{c^2 r^2}\vec{e_j})\frac{-GM}{c^2 r^2}\vec{e_i} = -\dot{r}\frac{G^2M^2}{c^4 r^4}\vec{e_i} \quad (72)$$

Collecting the terms I+II we get:

$$\frac{\ddot{\vec{v}}}{c^2} = \frac{1}{(1 - \frac{GM}{c^2 r})^2}\left[\left(2\dot{r}\frac{GM}{c^2 r^3} - 7\dot{r}\frac{G^2M^2}{c^4 r^4} + 4\dot{r}\frac{G^3M^3}{c^6 r^5}\right)\vec{e} + \left(-\dot{\varphi}\frac{GM}{c^2 r^2} + 3\dot{\varphi}\frac{G^2M^2}{c^4 r^3} - 2\dot{\varphi}\frac{G^3M^3}{c^6 r^4}\right)\vec{n}\right] \quad (73)$$

$$\frac{\ddot{\vec{v}}}{c^2} = \frac{1}{(1 - \frac{GM}{c^2 r})^2}\left[\left(-\frac{G\dot{M}}{c^2 r^2} + 7/3\frac{G^2\dot{M}^2}{c^4 r^3} - \frac{G^3\dot{M}^3}{c^6 r^4}\right)\vec{e} + \left(-\frac{GM}{c^2 r^2} + 3\frac{G^2M^2}{c^4 r^3} - 2\frac{G^3M^3}{c^6 r^4}\right)\dot{\varphi}\vec{n}\right] \quad (74)$$

Since $\dot{\varphi}\vec{n} = \dot{\vec{e}}$ we get:

$$\frac{\ddot{\vec{v}}}{c^2} = \frac{1}{\left(1 - \frac{GM}{c^2 r}\right)^2}\left[\left(\frac{-G\dot{M}}{c^2 r^2} + \frac{9}{3}\frac{G^2\dot{M}^2}{c^4 r^3} - \frac{2}{3}\frac{G^2\dot{M}^2}{c^4 r^3} - \frac{2G^3\dot{M}^3}{c^6 r^4} + \frac{G^3\dot{M}^3}{c^6 r^4}\right)\vec{e} + \left(\frac{-GM}{c^2 r^2} + 3\frac{G^2M^2}{c^4 r^3} - 2\frac{G^3M^3}{c^6 r^4}\right)\dot{\vec{e}}\right] \quad (75)$$

$$\frac{\ddot{\vec{v}}}{c^2} = \frac{1}{\left(1 - \frac{GM}{c^2 r}\right)^2}\frac{d}{dt}\left[\left(\frac{-GM}{c^2 r^2} + 3\frac{G^2M^2}{c^4 r^3} - 2\frac{G^3M^3}{c^6 r^4}\right)\vec{e}\right] + \frac{1}{\left(1 - \frac{GM}{c^2 r}\right)^2}(-2/3\frac{G^2\dot{M}^2}{c^4 r^3} + \frac{G^3\dot{M}^3}{c^6 r^4})\vec{e} \quad (76)$$

The last term in 76 can be shown to be equal to:

$$\left(-\frac{GM}{c^2r^2}+3\frac{G^2M^2}{c^4r^3}-2\frac{G^3M^3}{c^6r^4}\right)\vec{e}\,\frac{d}{dt}\frac{1}{(1-\frac{GM}{c^2r})^2}$$

Thus we finally get:

$$\frac{\dddot{\vec{v}}}{c^2}=\frac{d}{dt}\left[\frac{1}{(1-\frac{GM}{c^2r})^2}\left(\frac{-GM}{c^2r^2}+3\frac{G^2M^2}{c^4r^3}-2\frac{G^3M^3}{c^6r^4}\right)\vec{e}\right] \tag{77}$$

Integrating on time we get the gravitational acceleration: (the integration constant can be set to 0 in a suitable reference frame)

$$\frac{\dot{\vec{v}}}{c^2}=\frac{1}{(1-\frac{GM}{c^2r})^2}\left(-\frac{GM}{c^2r^2}+3\frac{G^2M^2}{c^4r^3}-2\frac{G^3M^3}{c^6r^4}\right)\vec{e} \tag{78}$$

To the first order in G we find Newton's law:

$$\vec{F}=m\dot{\vec{v}}=\frac{-GmM}{r^2}\vec{e} \tag{79}$$

and also the equivalent principle stating that the effects of a gravitational field are identical to an acceleration.

Solution in polar coordinates r, φ :

The radial $\vec{e}$ component of the acceleration is:

$$\frac{d^2r}{dt^2}-r(\frac{d\varphi}{dt})^2$$

And its tangential $\vec{n}$ component is:

$$r\frac{d^2\varphi}{dt^2}+2\frac{dr}{dt}\frac{d\varphi}{dt}=\frac{1}{r}\frac{d}{dt}(r^2\dot{\varphi})$$

We thus get the following set of equations:

$$\frac{d^2r}{dt^2}-r\left(\frac{d\varphi}{dt}\right)^2= c^2\left[\left(1-\frac{GM}{c^2r}\right)^{-2}\left(-\frac{GM}{c^2r^2}+3\frac{G^2M^2}{c^4r^3}-2\frac{G^3M^3}{c^6r^4}\right)\right] \tag{80}$$

And:

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\varphi})=0 \tag{81}$$

The equation 81 expresses the conservation of angular momentum and gives
$r^2\dot{\varphi}=h$ with h a constant (m^2/s)

Back to eq 80 , we must first develop $\frac{d^2r}{dt^2}$ as follows:

$$\frac{dr}{dt}=\frac{dr}{d\varphi}\frac{d\varphi}{dt}\rightarrow\dot{r}=\frac{h}{r^2}\frac{dr}{d\varphi}$$

$$\ddot{r}=\frac{\dot{r}}{d\varphi}\frac{d\varphi}{dt}=\frac{h^2}{r^4}\frac{d^2r}{d\varphi^2}-2\frac{h^2}{r^5}(\frac{dr}{d\varphi})^2 \tag{82}$$

The eq 80 becomes

$$\frac{h^2}{r^4}\frac{d^2r}{d\varphi^2}-2\frac{h^2}{r^5}(\frac{dr}{d\varphi})^2-\frac{h^2}{r^3}- \left(1-\frac{GM}{c^2r}\right)^{-2}\left(-\frac{GM}{r^2}+3\frac{G^2M^2}{c^2r^3}-2\frac{G^3M^3}{c^4r^4}\right)=0 \tag{83}$$

And after multiplying by r^2:

$$\frac{h^2}{r^2}\frac{d^2r}{d\varphi^2}-2\frac{h^2}{r^3}(\frac{dr}{d\varphi})^2-\frac{h^2}{r}- \left(1-\frac{GM}{c^2r}\right)^{-2}\left(-GM+3\frac{G^2M^2}{c^2r}-2\frac{G^3M^3}{c^4r^2}\right)=0 \tag{84}$$

Let $u=\frac{1}{r}$ then $\frac{du}{d\varphi}=-\frac{1}{r^2}\frac{dr}{d\varphi}$
and $\frac{d^2u}{d\varphi^2}=-\frac{1}{r^2}\frac{d^2r}{d\varphi^2}+\frac{2}{r^3}(\frac{dr}{d\varphi})^2$
We get:

$$-h^2\frac{d^2u}{d\varphi^2}-h^2u+ \left(1-\frac{GMu}{c^2}\right)^{-2}\left(GM-\frac{3G^2M^2u}{c^2}+\frac{2G^3M^3u^2}{c^4}\right)=0 \tag{85}$$

Or:

$$-h^2\frac{d^2u}{d\varphi^2}-h^2u+ GM\left(\frac{1-\frac{2GMu}{c^2}+\frac{G^2M^2u^2}{c^4}-\frac{GMu}{c^2}+\frac{G^2M^2u^2}{c^4}}{1-\frac{2GMu}{c^2}+\frac{G^2M^2u^2}{c^4}}\right)=0$$

Dividing by h^2, writing $\frac{du}{d\varphi}\equiv u'$ and rearranging we get:

$$u''+u-\frac{GM}{h^2}\left(1-\frac{\frac{GMu}{c^2}}{1-\frac{GMu}{c^2}}\right)=0$$

Or again:

$$u''+u\left(1+\frac{G^2M^2}{c^2h^2}\left(1-\frac{GMu}{c^2}\right)^{-1}\right)=\frac{GM}{h^2} \tag{86}$$

This is the equation of the orbit of a body around a stationary body of mass M for non-relativistic speed.

Relativistic Equations of motion of a body in a central static field:

We rewrite the equation 61 in which we will use the relativistic impulsion $\vec{p}$ and p_0 :

$$\ddot{p_i} = \partial_i \left[p_0 v_j \frac{1-2\Gamma}{1-\Gamma} \partial_j \Gamma \right] \tag{87}$$

With the relativistic impulsion of a body according to [1]

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{88}$$

and

$$p_0 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{89}$$

One gets:

$$\ddot{p_i} = p_0 v_j \partial_i \left[\frac{1-2\Gamma}{1-\Gamma} \partial_j \Gamma \right] + \left(\frac{1-2\Gamma}{1-\Gamma} \partial_j \Gamma \right) v_j \partial_i p_0 \tag{90}$$

$$\ddot{p_i} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \frac{d}{dt} \left[\frac{1-2\Gamma}{1-\Gamma} \partial_j \Gamma \right] + \left[\frac{1-2\Gamma}{1-\Gamma} \partial_j \Gamma \right] \delta_i^j \frac{d}{dt} p_0 \tag{91}$$

$$\ddot{p_i} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \frac{d}{dt} \left[\frac{1-2\Gamma}{1-\Gamma} \partial_j \Gamma \right] + \left[\frac{1-2\Gamma}{1-\Gamma} \partial_j \Gamma \right] \delta_i^j \frac{d}{dt} \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{92}$$

In polar coordinates we had calculated that:

$$\frac{1-2\Gamma}{1-\Gamma} \partial_j \Gamma = \frac{1}{\left(1-\frac{GM}{c^2 r}\right)^2} \left(-\frac{GM}{c^2 r^2} + 3\frac{G^2 M^2}{c^4 r^3} - 2\frac{G^3 M^3}{c^6 r^4} \right) \vec{e_j}$$

Replacing we eventually get:

$$\ddot{\vec{p}} = \frac{d}{dt} \left[\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \frac{1}{\left(1-\frac{GM}{c^2 r}\right)^2} \left(\frac{-GM}{c^2 r^2} + 3\frac{G^2 M^2}{c^4 r^3} - 2\frac{G^3 M^3}{c^6 r^4} \right) \vec{e} \right]$$

Integrating on time we get:

$$\dot{\vec{p}} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \frac{1}{\left(1-\frac{GM}{c^2 r}\right)^2} \left(\frac{-GM}{c^2 r^2} + 3\frac{G^2 M^2}{c^4 r^3} - 2\frac{G^3 M^3}{c^6 r^4} \right) \vec{e} \tag{93}$$

Let us now evaluate the time derivative of the impulsion:

$$\frac{d}{dt}\vec{p} = m \left(\frac{\dot{\vec{v}}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{(\vec{v}|\dot{\vec{v}})\vec{v}}{c^2(1-\frac{v^2}{c^2})^{3/2}} \right) \tag{94}$$

Note that according to [1], when the force is normal to speed we get:

$$\frac{d}{dt}\vec{p} = m \left(\frac{\dot{\vec{v}}}{\sqrt{1-\frac{v^2}{c^2}}} \right) \tag{95}$$

And when the force and speed are co-linear we get:

$$\frac{d}{dt}\vec{p} = m \left(\frac{\dot{\vec{v}}}{(1-\frac{v^2}{c^2})^{3/2}} \right) \tag{96}$$

In the following, since forces and speeds can have independent orientations we have to use the general formula 94 for the time derivative of impulsion.
In polar coordinates we have:

$$\dot{\vec{v}} = (\ddot{\vec{r}} - r\dot{\varphi}^2)\vec{e} + \frac{1}{r}(\dot{r^2\dot{\varphi}})\vec{n}$$

and:

$$(\vec{v}|\dot{\vec{v}}) = \frac{1}{2}\dot{v}^2 = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2)$$

Then the equations 93 and 94 result in:
Along $\vec{n}$:

$$\frac{\frac{1}{r}(\dot{r^2\dot{\varphi}})}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{r\dot{\varphi}(\dot{v^2})}{2c^2(1-\frac{v^2}{c^2})^{3/2}} = 0 \tag{97}$$

And along $\vec{e}$: (dividing both members by m)

$$\frac{\ddot{r} - r\dot{\varphi}^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{\dot{r}\,\dot{v^2}}{2c^2(1-\frac{v^2}{c^2})^{3/2}} = \frac{c^2}{\sqrt{1-\frac{v^2}{c^2}}} \frac{1}{\left(1-\frac{GM}{c^2 r}\right)^2} \left(\frac{-GM}{c^2 r^2} + 3\frac{G^2 M^2}{c^4 r^3} - 2\frac{G^3 M^3}{c^6 r^4} \right) \tag{98}$$

The equation 97 will give us the relativistic conservation of angular momentum, after the following development:
Multiply both members by $\sqrt{1-\frac{v^2}{c^2}}$:
$\Rightarrow (\dot{r^2\dot{\varphi}}) = -r^2\ \dot{\varphi}\ \frac{\dot{v^2}}{2(c^2-v^2)}$
Thus:

$$\frac{(\dot{r^2\dot{\varphi}})}{r^2\dot{\varphi}} = -\frac{\dot{v^2}}{2(c^2-v^2)} = \frac{1}{2}\frac{(\dot{c^2 - v^2})}{(c^2-v^2)} \tag{99}$$

Integrating on time we get:

$$\ln(r^2\dot{\varphi}) = \frac{1}{2}\ln(c^2 - v^2) + \kappa \quad (100)$$

with κ a constant
and after exponentiation and defining $e^\kappa \equiv h/c$

$$r^2\dot{\varphi} = \sqrt{(c^2 - v^2)}\ e^\kappa = \sqrt{(c^2 - v^2)}\ h/c = h\sqrt{1 - \frac{v^2}{c^2}} \quad (101)$$

For $v \ll c$ one gets the classical form $r^2\dot{\varphi} = h$ however for the calculation of the precession of the perihelion of Mercury we will use the equation 101

Let us now develop the equation 98 that will give us the relativistic relation for the orbit of a body around a stationary body of mass M :
After multiplication of both members of 98 by $\sqrt{1 - \frac{v^2}{c^2}}$ we get:

$$\ddot{r} - r\dot{\varphi}^2 + \frac{\dot{r}(\dot{v^2})}{2(c^2 - v^2)} = c^2\frac{1}{(1 - \frac{GM}{c^2 r})^2}\left(\frac{-GM}{c^2r^2} + 3\frac{G^2M^2}{c^4r^3} - 2\frac{G^3M^3}{c^6r^4}\right) \quad (102)$$

We now expand the first member of 102 so that r depends on φ, using the relations $\dot{\varphi} = \frac{h}{r^2}\sqrt{1 - \frac{v^2}{c^2}}$ and $\dot{r} = \frac{dr}{d\varphi}\dot{\varphi} = r'\frac{h}{r^2}\sqrt{1 - \frac{v^2}{c^2}}$
Calculation of $\ddot{r}$:

$$\ddot{r} = \frac{d\dot{r}}{d\varphi}\dot{\varphi} = \frac{h}{r^2}\sqrt{1 - \frac{v^2}{c^2}}\left(\frac{d}{d\varphi}(\frac{h}{r^2}\sqrt{1 - \frac{v^2}{c^2}}\frac{dr}{d\varphi})\right) = \frac{h}{r^2}\sqrt{1 - \frac{v^2}{c^2}}\left(\frac{-2}{r^3}r'^2\sqrt{1 - \frac{v^2}{c^2}} + \frac{1}{r^2}\frac{d}{d\varphi}\sqrt{1 - \frac{v^2}{c^2}} + \frac{\sqrt{1 - \frac{v^2}{c^2}}}{r^2}r''\right)$$

$$\ddot{r} = \frac{h}{r^2}\left(\frac{-2}{r^3}r'^2(1 - \frac{v^2}{c^2}) + \frac{r'}{r^2}\sqrt{1 - \frac{v^2}{c^2}}\frac{d}{d\varphi}\sqrt{1 - \frac{v^2}{c^2}} + \frac{1 - \frac{v^2}{c^2}}{r^2}r''\right) \quad (103)$$

Now $\sqrt{1 - \frac{v^2}{c^2}}\frac{d}{d\varphi}\sqrt{1 - \frac{v^2}{c^2}} = 1/2\frac{d}{d\varphi}(1 - \frac{v^2}{c^2}) = 1/2\frac{d}{d\varphi}(-\frac{v^2}{c^2})$

And $\frac{d}{d\varphi}v^2 = \dot{v^2}\frac{dt}{d\varphi} = \dot{v^2}\frac{r^2}{h\sqrt{1 - \frac{v^2}{c^2}}}$

Replacing in 103 we get:

$$\ddot{r} = (1 - \frac{v^2}{c^2})(\frac{-2h^2r'^2}{r^5}) - \frac{hr'\dot{v^2}}{2r^2c^2\sqrt{1 - \frac{v^2}{c^2}}} + (1 - \frac{v^2}{c^2})\frac{h^2r''}{r^4} \quad (104)$$

The two other terms are:

$$-r\dot{\varphi}^2 = \frac{-h^2}{r^3}(1 - \frac{v^2}{c^2}) \quad (105)$$

And:

$$\frac{\dot{r}\dot{v^2}}{2c^2(1 - \frac{v^2}{c^2})} = \frac{hr'\dot{v^2}}{2r^2c^2\sqrt{1 - \frac{v^2}{c^2}}} \quad (106)$$

via: $\dot{r} = \frac{dr}{d\varphi}\frac{d\varphi}{dt}$ and $\dot{\varphi} = \frac{h}{r^2\sqrt{1 - \frac{v^2}{c^2}}}$
So the first member $\ddot{r} - r\dot{\varphi}^2 + \frac{\dot{r}(\dot{v^2})}{2(c^2 - v^2)}$ becomes:

$$(1 - \frac{v^2}{c^2})(\frac{-2h^2r'^2}{r^5} + \frac{h^2r''}{r^4} - \frac{h^2}{r^3}) \quad (107)$$

And dividing both members of 102 by $(1 - \frac{v^2}{c^2})$ we get:

$$\frac{-2h^2r'^2}{r^5} + \frac{h^2r''}{r^4} - \frac{h^2}{r^3} = \frac{1}{(1 - \frac{v^2}{c^2})\,(1 - \frac{GM}{c^2r})^2}\left(\frac{-GM}{r^2} + 3\frac{G^2M^2}{c^2r^3} - 2\frac{G^3M^3}{c^4r^4}\right) \quad (108)$$

We now apply the same mathematical treatment to 108 as for the equation 84:

After multiplying by r^2:

$$\frac{-2h^2r'^2}{r^3} + \frac{h^2r''}{r^2} - \frac{h^2}{r} - \frac{1}{(1 - \frac{v^2}{c^2})\,(1 - \frac{GM}{c^2r})^2}\left(-GM + 3\frac{G^2M^2}{c^2r} - 2\frac{G^3M^3}{c^4r^2}\right) = 0 \quad (109)$$

Let $u = \frac{1}{r}$ then $\frac{du}{d\varphi} = -\frac{1}{r^2}\frac{dr}{d\varphi}$
and $\frac{d^2u}{d\varphi^2} = -\frac{1}{r^2}\frac{d^2r}{d\varphi^2} + \frac{2}{r^3}(\frac{dr}{d\varphi})^2$

We get the following differential equation in u:

$$-h^2\frac{d^2u}{d\varphi^2} - h^2u + \left(1 - \frac{GMu}{c^2}\right)^{-2}(1 - \frac{v^2}{c^2})^{-1}\left(GM - \frac{3G^2M^2u}{c^2} + \frac{2G^3M^3u^2}{c^4}\right) = 0 \quad (110)$$

Or:

$$-h^2\frac{d^2u}{d\varphi^2}-h^2u+(1-\frac{v^2}{c^2})^{-1}$$
$$GM\left(\frac{1-\frac{2GMu}{c^2}+\frac{G^2M^2u^2}{c^4}-\frac{GMu}{c^2}+\frac{G^2M^2u^2}{c^4}}{1-\frac{2GMu}{c^2}+\frac{G^2M^2u^2}{c^4}}\right)=0$$

Dividing by h^2, writing $\frac{du}{d\varphi}\equiv u'$ and rearranging we get:

$$u''+u-\frac{GM}{h^2(1-\frac{v^2}{c^2})}\left(1-\frac{\frac{GMu}{c^2}}{1-\frac{GMu}{c^2}}\right)=0 \quad (111)$$

Or else:

$$u''+u\left(1+\frac{G^2M^2}{c^2h^2(1-\frac{v^2}{c^2})(1-\frac{GMu}{c^2})}\right) = \frac{GM}{h^2(1-\frac{v^2}{c^2})} \quad (112)$$

This is the polar equation of the orbit of a body (a planet) around a stationary body of mass M for relativistic speed.
It contains factors $(1-\frac{v^2}{c^2})$ which depend on u.
Let us find a relation between $(1-\frac{v^2}{c^2})$ and u:
We observe that both $1/(1-\frac{v^2}{c^2})$ and u are periodic in φ on a classical non relativistic orbit, see figures 10 and 11.

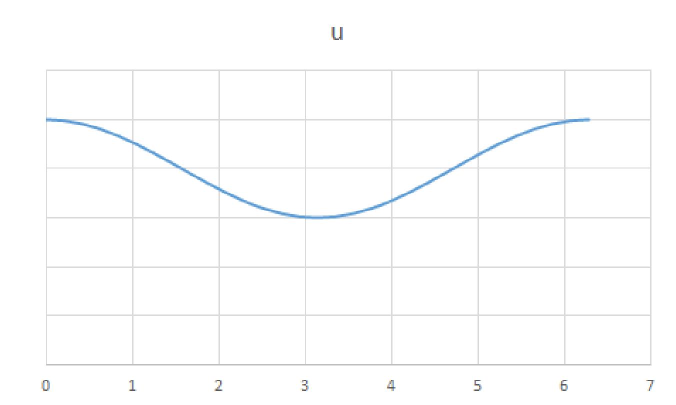

Figure 10. $u(\varphi)$

We define the relation between $1/(1-\frac{v^2}{c^2})$ and u as:

$$\frac{1}{(1-\frac{v^2}{c^2})}=A\frac{u}{(\frac{GM}{h^2})}+B \quad (113)$$

where u is made non dimensional via the factor $1/\frac{GM}{h^2}$.
A and B can be derived from the values of $1/(1-\frac{v^2}{c^2})$ and u at the perihelion and aphelion of the planet, the result is, with e being the eccentricity:

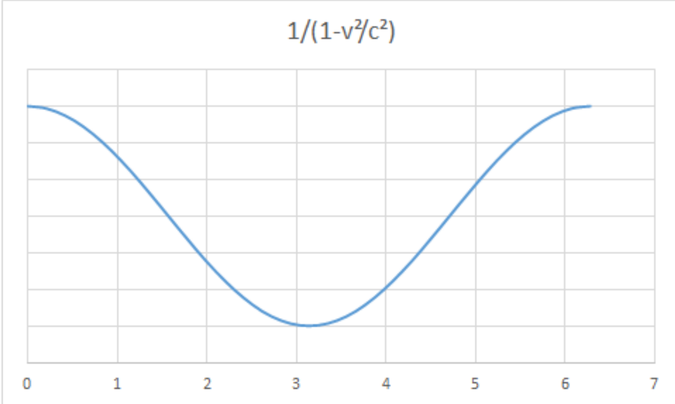

Figure 11. $1/(1-\frac{v^2}{c^2})(\varphi)$

$A=\frac{2G^2M^2}{c^2h^2}$ and $B=1-\frac{G^2M^2(1-e^2)}{c^2h^2}$

And $1/(1-\frac{v^2}{c^2})$ is:

$1/(1-\frac{v^2}{c^2})=(\frac{2GMu}{c^2}+1-\frac{G^2M^2(1-e^2)}{c^2h^2})$

The equation 112 then becomes:

$$u''+u\left(1+\frac{G^2M^2}{c^2h^2(1-\frac{GMu}{c^2})}(\frac{2GMu}{c^2}+1-\frac{G^2M^2(1-e^2)}{c^2h^2})\right) = \frac{2G^2M^2u}{c^2h^2}+\frac{GM}{h^2}-\frac{G^3M^3(1-e^2)}{c^2h^4}) \quad (114)$$

VIII. PRECESSION OF THE PERIHELION OF MERCURY

Let us find an approximate solution to the equation 114 to the orbit of Mercury around the Sun in which the term $(\frac{1}{1-\frac{GMu}{c^2}})$ is approximated by $1+\frac{GMu}{c^2}$, since $\frac{GMu}{c^2}\simeq 2.5\,10^{-8}$
And 114 becomes, with $1-\frac{G^2M^2(1-e^2)}{c^2h^2}$ written B for brevity:

$$u''+u\left(1+\frac{G^2M^2}{c^2h^2}(1+\frac{GMu}{c^2})(\frac{2GMu}{c^2}+B)\right) = \frac{2G^2M^2u}{c^2h^2}+\frac{GM}{h^2}B \quad (115)$$

$$u''+u\left(1+\frac{G^2M^2}{c^2h^2}(B+\frac{GMu}{c^2}(2+B)+\frac{2G^2M^2u^2}{c^4})\right) = \frac{2G^2M^2u}{c^2h^2}+\frac{GM}{h^2}B \quad (116)$$

$$u'' + u\Bigg(1 - \frac{2G^2M^2u}{c^2h^2} + \frac{G^2M^2}{c^2h^2}$$
$$(1 - \frac{G^2M^2(1-e^2)}{c^2h^2} + \frac{GMu}{c^2}(2+B) + \frac{2G^2M^2u^2}{c^4})\Bigg) = \frac{GM}{h^2}B \quad (117)$$

Replacing B by its value in the first term we get:

$$u'' + u(1 - \frac{2G^2M^2}{c^2h^2} - \frac{G^4M^4(1-e^2)}{c^4h^4}) + u^2(\frac{3G^3M^3}{c^4h^2}) + u^3(\frac{2G^4M^4}{c^6h^2}) = \frac{GM}{h^2}B \quad (118)$$

We now look at a solution of the homogeneous equation in the form $u = \alpha\cos(\beta\varphi)$ with $\alpha = \frac{eGM}{h^2}$
We have: $u'' = -\beta^2\alpha\cos(\beta\varphi)$
$u^2 = \alpha^2\cos^2(\beta\varphi) = \frac{\alpha^2}{2}\cos(2\beta\varphi) + \frac{\alpha^2}{2}$
$u^3 = \alpha^3\cos^3(\beta\varphi) = \frac{\alpha^3}{4}\cos(3\beta\varphi) + \frac{3\alpha^3}{4}\cos(\beta\varphi)$
Replacing and solving for the term $\cos(\beta\varphi)$ we get:

$$-\beta^2\alpha\cos(\beta\varphi) + (1 - \frac{G^2M^2}{c^2h^2})\alpha\cos(\beta\varphi) + \frac{2G^4M^4}{c^6h^2}\frac{3\alpha^3}{4}\cos(\beta\varphi) = 0 \quad (119)$$

or:

$$-\beta^2 + (1 - \frac{G^2M^2}{c^2h^2}) + \frac{2G^4M^4}{c^6h^2}\frac{3\alpha^2}{4} = 0 \quad (120)$$

thus:

$$\beta = \sqrt{1 - \frac{G^2M^2}{c^2h^2} + 3/2e^2(\frac{G^2M^2}{c^2h^2})^3} \quad (121)$$

The last term $3/2e^2(\frac{G^2M^2}{c^2h^2})^3$ is negligible and we get for the homogeneous solution:

$$u = \alpha\cos\left(\sqrt{1 - \frac{G^2M^2}{c^2h^2}}\,\varphi\right) \simeq \alpha\cos\left[\left(1 - \frac{G^2M^2}{2c^2h^2}\right)\varphi\right] \quad (122)$$

since $\frac{G^2M^2}{c^2h^2} \simeq 2.6\,10^{-8} \ll 1$

The complete solution to 118 is:

$$u = \frac{eGM}{h^2}\cos\left[\left(1 - \frac{G^2M^2}{2c^2h^2}\right)\varphi\right] + \frac{GM}{h^2}(1 - \frac{G^2M^2(1-e^2)}{c^2h^2}) \quad (123)$$

Advance of the perihelion:
With $M = 2\,10^{30}\,kg$ the mass of the sun
$G = 6.67\,10^{-11} m^3/kgs^2$
$r = 1/u = 57.9\,10^9 m$ is the average distance of Mercury to the Sun
$e = 0.204$ for Mercury
$h = 2.7\,10^{15} m^2/s$ for Mercury
$c = 3\,10^8 m/s$
The period of revolution of Mercury = 88 days
We get a phase shift due to the $\frac{G^2M^2}{c^2h^2}$ term in the orbit equation 123.

This shift is $\frac{2\pi G^2M^2}{2c^2h^2}$ radians per revolution.

This corresponds to 7.2" per century. The corresponding change in the position of the perihelion moves forward to the orbit of Mercury, the accepted value up to now is 43" in the same direction. An explanation for this discrepancy deserves to be investigated.[14]

IX. BLACK HOLES

Objects whose gravitational fields are too strong for light to escape were already considered in the 18th century by John Michell and Pierre-Simon Laplace. When described by general relativity, the black hole contains a gravitational singularity at the origin, a region where the spacetime curvature becomes infinite and contains all the mass of the black hole.

We will first limit ourselves to the study of black holes having mass M but neither electric charge nor angular momentum.
What happens to a body in the vicinity r of a black hole? Will it be swallowed and disappear for ever ?
Let r be the distance of this object to the centre of the black hole, which is supposed to be located at r=0 and to contain the mass M. We don't know yet if the large body of mass M is a black hole or not, at this point it is just a homogeneous body with the mass M "concentrated" at the origin. The radial extension of the massive body doesn't matter as long as it is smaller than or equal to r, because if r is smaller than the radial extension of the large body of mass M, then some mass will not be taken into account when calculating the force of attraction at radius r. When assuming r to be the radial extension of the body of mass M, the following equations express the force acting on the surface of the body, i.e. at radius r. The equations 80 and 81 describe the motion of a body in a central static field. The gravitational acceleration is 78:

$$c^2\left(1 - \frac{GM}{rc^2}\right)^{-2}\left(-\frac{GM}{r^2c^2} + 3\frac{G^2M^2}{c^4r^3} - 2\frac{G^3M^3}{c^6r^4}\right)$$

This is the radial force $F(r)$ acting on a unit mass, unit [N]
We choose the unit system $\frac{GM}{c^2} = 1$ in the following calculation, for brevity. Then

$$F(r) = (1 - \frac{1}{r})^{-2}(-\frac{1}{r^2} + \frac{3}{r^3} - \frac{2}{r^4}) \quad (124)$$

has two singularities at r=0 and 1 and a zero at r=2. Between r=0 and r= $\frac{GM}{c^2}$ the force is always attractive ($F < 0$)

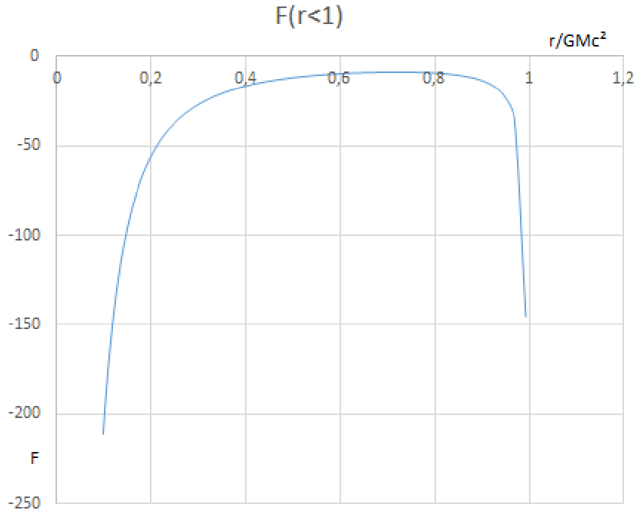

Figure 12. F for $r < GM/c^2$

The region from r=0 to $r = \frac{GM}{c^2}$ is attractive, with infinite attraction for r=0 and r= $\frac{GM}{c^2}$.
Between $r = \frac{GM}{c^2}$ and $r = 2\frac{GM}{c^2}$ the force is repulsive ($F > 0$) and is attractive again for $r > 2\frac{GM}{c^2}$
The force then follows a $\frac{1}{r^2}$ law for large r.

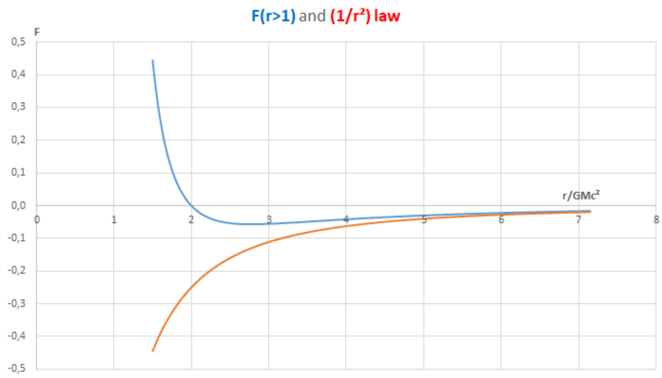

Figure 13. F for $r > GM/c^2$

The point $2\frac{GM}{c^2}$ is stable in equilibrium with zero force, it is called the limit radius R_L.
Incidentally it is equal to the Schwarzschild horizon in GR.
The black hole can never shrink to a null radius: an infinite repulsive barrier at r= $\frac{GM}{c^2}$ prevents this collapse to happen.
R_L represents the stable size of a non-rotating black hole. At R_L, the surface of the black hole is at equilibrium, no force acting on the surface.
The force 78 derives from the following potential $\Phi(r)$, for $r > \frac{GM}{c^2}$ and set to zero at r_∞:

$$\Phi(r) = c^2\left(\frac{2GM}{rc^2} + \ln\left(1 - \frac{GM}{rc^2}\right)\right)\left[\frac{m^2}{s^2}\right] \qquad (125)$$

Which can be developed in the following series:

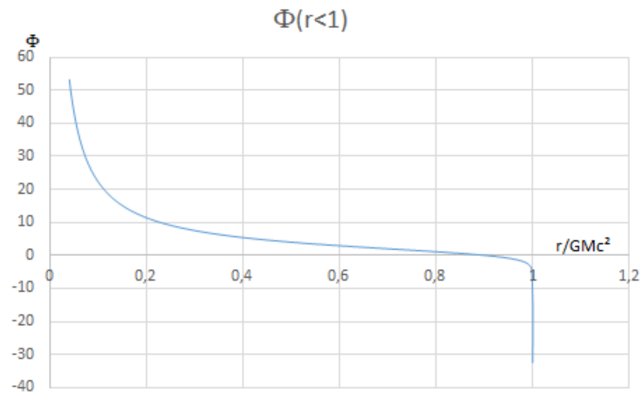

Figure 14. Φ for $r < GM/c^2$

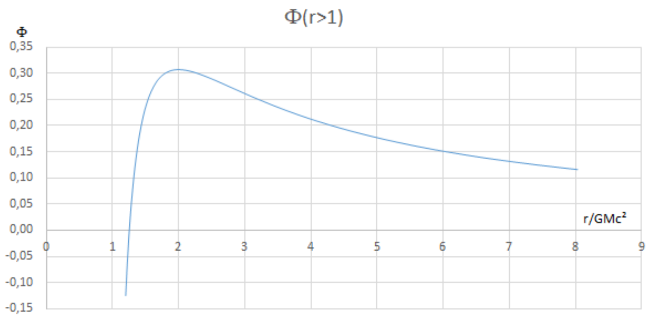

Figure 15. Φ for $r > GM/c^2$

$$\Phi(r) = \frac{GM}{r} - \frac{G^2M^2}{2r^2c^2} - \frac{G^3M^3}{3r^3c^4} - \ldots \qquad (126)$$

This is the sum of an attractive potential $\frac{GM}{r}$ and a repulsive potential equal to $-\frac{G^2M^2}{2r^2c^2} - \frac{G^3M^3}{3r^3c^4} - \ldots$

The latter repulsive term can be seen as a graviton-graviton interaction term: it is negligible for large distance r but predominant at short distances. It has a maximum of $c^2(1 + \ln(1/2)) \simeq 0.307\,c^2$ at R_L.

The escape of light from the "black hole"

(Following a classical approach equaling potential and kinetic energy)
Here we plot $-\Phi$ for a more intuitive understanding.
We have for a unit mass m:

$$1/2\,mv^2 = mc^2\left(\frac{2GM}{rc^2} + \ln\left(1 - \frac{GM}{rc^2}\right)\right)$$

Dividing by m and equalling v to c:

$$1/2 = \left(\frac{2GM}{rc^2} + \ln\left(1 - \frac{GM}{rc^2}\right)\right)$$

There is no solution to this equation, the kinetic energy

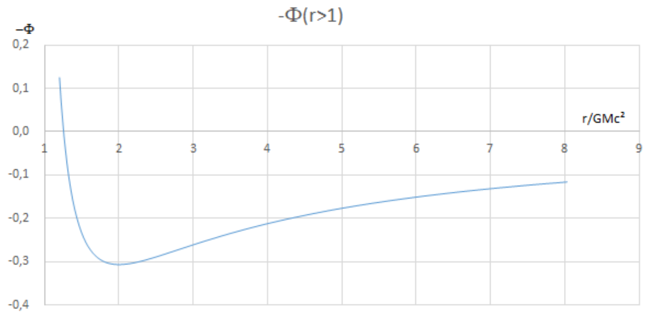

Figure 16. $-\Phi$ for $r > GM/c^2$

is always higher than the potential well, so light can always escape from the "black hole".
We need to find a new name for this object of radius R_L: **it is not a hole and it is not black.**
We propose the name CORE, which has the same etymology as heart and can also mean nucleus.
The core radius of stability R_L is equal to the Schwarzschild radius of GR. The core sits there in stable equilibrium between expansion and contraction forces.
If there was no repulsive terms: $-\frac{G^2M^2}{2r^2c^2} - \frac{G^3M^3}{3r^3c^4} - \ldots$ in the potential, then a solution would be possible where no light can escape, at a horizon radius equal to the Schwarzschild radius.

X. FREQUENCY SHIFT OF A CORE

The core is still considered as a non-rotating mass M. From equation 1 we have, taking the potential Φ into account:

$$\nu_1 \left(1 - \frac{\Phi_1}{c^2}\right) = \nu_2 \left(1 - \frac{\Phi_2}{c^2}\right)$$

If the position 1 is at infinite distance $r = \infty$, then $\Phi_1 = 0$
At position 2, $r = R_L$ and $\Phi_2 = 0.307c^2$
Then $\nu_1 = \nu_2(1 - 0.307) = 0.693\,\nu_2$
The frequency of a spectral line emitted from a point at the surface R_L of a core will be perceived from an infinite distance at 0.693 times the frequency due to time slowdown at the surface of the core.
But if we take into account the potential well of $0.307\,c^2$, the emitted frequency would decrease by a factor of 0.614 since $E = h\nu$ and E is reduced by a factor $\frac{0.307}{((c^2)/2)}$.
The total frequency shift factor is the 0.693 times (1-0.614) = 0.268.

XI. PULSATION OF A CORE

On the surface of the core, a mass is at equilibrium but can also oscillate radially around the equilibrium point R_L. Let us look at its first mode of oscillation: For a unit mass m=1 on the surface of the core, the oscillation frequency ω is $\sqrt{\frac{k}{m}} = \sqrt{k}$
And

$$k = \frac{dF}{dr}|_{R_L} = -\frac{c^6}{4G^2M^2} \tag{127}$$

$$\sqrt{|k|} = \frac{c^3}{2GM}$$

Thus $\omega = \frac{c^3}{2GM}$ for a core of mass M
For instance a non-rotating core of mass = 1000 times the sun mass will pulsate on it's first mode at $\omega = 101\,rad/s = 16Hz$.

XII. EXPANSION OF THE UNIVERSE

The estimated mass of the known universe is in a range of $1.7\,10^{52}$ to $1.7\,10^{54}kg$.
Then let us calculate the R_L of the universe:
$R_L = \frac{2GM}{c^2} = 2.5\,10^{25}$ to $2.5\,10^{27}\,m$
The estimated radius of the universe according to the standard cosmological model is $46\,10^9$ light years = $4.2\,10^{26}\,m$.
So the estimated radius of the universe is in the same range of magnitude as its R_L radius and it could even be very close to its R_L radius ! And the universe could then be considered as a core.
Here is a figure 17 showing $-\Phi$ of the universe between 0.7 and $2\,R_L$.

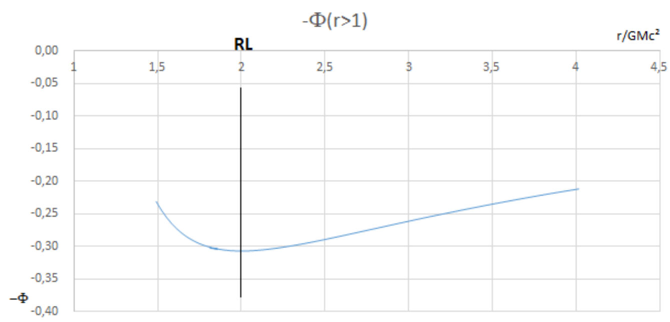

Figure 17. $-\Phi$ for $r > GM/c^2$

Considered as a non rotating core, the universe pulsates around its R_L size, at a frequency $\frac{c^3}{2GM}$ or a period $\frac{\frac{4\pi c^3}{2GM}}{\frac{2GM}{c^2}} = 278\,10^9$ years, considering $R_L = 4.2\,10^{26}m =$ (point 2 on the graph 17)
If the universe is now in an expansion phase, this would

mean that its size is presently lower than its horizon R_L.
If the universe had begun at a size $< 1.27\,R_L\,/2$, it would have enough potential energy to expand to an infinite radius if there is no energy loss during that expansion.
Otherwise the universe will oscillate or fluctuate around its R_L radius, where Φ is maximum.
This is reminiscent of A.D.Sakharov's concept of the fluctuating or oscillating universe.[20][21]

XIII. ACCELERATION OF THE EXPANSION OF THE UNIVERSE

Let us consider a body of mass m situated on the rim R_L of the universe, the force acting on this body is 78 :

$$F = m\,c^2 \frac{1}{(1-\frac{GM}{c^2 r})^2}\left(\frac{-GM}{c^2 r^2} + 3\frac{G^2M^2}{c^4 r^3} - 2\frac{G^3M^3}{c^6 r^4}\right)$$

The equation of motion is

$$m\,\ddot{r} - c^2 \frac{1}{(1-\frac{GM}{c^2 r})^2}\left(\frac{-GM}{c^2 r^2} + 3\frac{G^2M^2}{c^4 r^3} - 2\frac{G^3M^3}{c^6 r^4}\right) m = 0$$

With the potential that is

$$\Phi(r) = c^2\left(\frac{2GM}{rc^2} + \ln\left(1 - \frac{GM}{rc^2}\right)\right)$$

We represent $-\Phi$ on the graph, it is more intuitive.

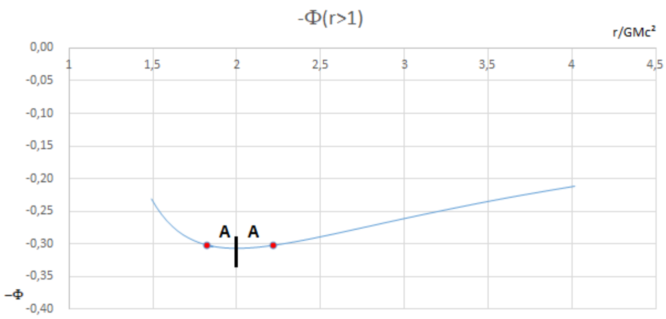

Figure 18. Amplitude A

For small motion around R_L, we approximate $F(r)$ by an harmonic force $F(r) = k.(r - RL)$
With $k = \frac{c^6}{4G^2M^2}$ (eq 127)
It results, with $r = R_L$ at $t = 0$ we get:

$$(r - R_L) = A\sin\left(\frac{c^3}{GM}t\right)$$

A being the maximal fluctuation in size of the universe

$$(r \dot{-} R_L) = A\frac{c^3}{GM}\cos\left(\frac{c^3}{GM}t\right)$$

is is the expansion rate of the universe

$$(r \ddot{-} R_L) = -A\frac{c^6}{G^2M^2}\sin\left(\frac{c^3}{GM}t\right)$$

is the acceleration of the expansion of the universe

When $r < R_L$, the expansion and the acceleration are both positive. When the radius of the universe will reach its horizon R_L the expansion will continue but at a decelerating rate until the universe reaches its maximum size $R_L + A$ and stops expanding. Then the inverse movement will take place.
A and t are two unknowns which can be determined by the values of the expansion rate (Hubble-Lemaître constant = 70 km/s/Mps) and the acceleration of the universe expansion.

XIV. CONCLUSION

The present model for gravitation is not equivalent to general relativity in that the space is flat with no curvature.
The existence of a short range repulsive potential leads to very different results, especially with regard to black holes which become "cores" with a finite size.
The current that generates the gravitational forces is the Energy Impulsion tensor and this is a natural consequence of the invariance of the action under the group of translations in space-time.
The present theory has an abelian group of symmetry, like electromagnetism (U_1)[1]. However in electromagnetism the field A_μ is added to the impulsion p^μ in the Lagrangian, while in gravitation the field Γ^ν_μ is multiplied by the impulsion p^μ in the Lagrangian.
This is the origin of the short distance gravitational repulsive term and of the difference between gravitational and electromagnetic force.
The repulsive$-\frac{G^2M^2}{r^2}$ potential term is absent from general relativity. For that reason and in order to explain the acceleration of the expansion of the universe, the influence of a hypothetical 'dark energy' was invoked in GR.
Our model doesn't need 'dark energy' to explain the acceleration of the expansion of the universe.
However the repulsive $-\frac{G^2M^2}{r^2}$ potential also tends to open the orbit of Mercury and limit the advance the perihelion to 7.25" per century. This discrepancy with the measured value of 43" needs further investigation.
The expansion of Universe is a consequence of the universe being considered as a core with its natural pulsation frequency of one cycle per $278\,10^9$ years. As such the universe would radially oscillate around an equilibrium point.
The speed distribution in rotating galaxies arms could also be calculated in the new theoretical model, taking into account a "Lorentz" force. What's more, we can show that this "Lorentz" force acts in the right centripetal direction without maybe having to rely on "dark

matter" to do the job.
Quantum gravity has the massless spin 2 graviton for propagator of the interaction and this quantification could be the subject of further work.
Rotating cores are also a topic for further study.

XV. ACKNOWLEDGMENTS

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